7. Dirac’s δ-function

Consider the function
\[ \delta_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{x^2}{2\sigma^2} \right), \quad x \in \mathbb{R} \]  \hfill (1)

(a) Calculate \( \int_{\mathbb{R}} dx \delta_\sigma(x) \) and plot \( \delta_\sigma(x) \) as a function of \( x \) for various values of \( \sigma > 0 \).

(b) Calculate \( \tilde{\delta}(x) := \lim_{\sigma \to 0} \delta_\sigma(x) \) and \( \int_{\mathbb{R}} dx \tilde{\delta}(x) := \lim_{\sigma \to 0} \int_{\mathbb{R}} dx \delta_\sigma(x) \). Plot \( \tilde{\delta}(x) \) in the diagram of part (a). Why is \( \tilde{\delta} \) not a function?

(c) “Dirac’s δ-function” \( \delta(x) \) is understood as a shorthand notation for \( \lim_{\sigma \to 0} \delta_\sigma(x) \), provided the limit \( \sigma \to 0 \) is taken at the end of the calculation. Accordingly, calculate
\[ \int_{\mathbb{R}} dx \delta(x - y)f(x) = \lim_{\sigma \to 0} \int_{\mathbb{R}} dx \delta_\sigma(x - y)f(x), \quad y \in \mathbb{R}, \quad f : \mathbb{R} \to \mathbb{R} \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function (typically called “test function”) which for \( |x| \to \infty \) does not (or does sufficiently weakly) diverge. How is Dirac’s δ-function related to \( \tilde{\delta} \) considered in (b)?

(d) Proof the following relations:
\begin{enumerate}
    \item \( f(x)\delta(x - y) = f(y)\delta(x - y) \), \quad x,y \in \mathbb{R}, \quad f : \mathbb{R} \to \mathbb{R} \)
    \item \( \left( \frac{d^k}{dx^k} \delta(x) \right) = (-1)^k \delta(x) - \frac{d^k}{dx^k} \delta(x) \), \quad x \in \mathbb{R}, \quad k \in \mathbb{N} \)
    \item \( \int_{\mathbb{R}} dy \delta(x - y)\delta(y - z) = \delta(x - z) \), \quad x,z \in \mathbb{R} \)
    \item \( \delta(cx) = \frac{1}{|c|} \delta(x) \), \quad x \in \mathbb{R}, \quad c \in \mathbb{R} \setminus \{0\} \)
    \item \( \delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \), assuming the function \( f : \mathbb{R} \to \mathbb{R} \) has only zeros \( x_i \in \mathbb{R} \) of first order.
\end{enumerate}

(e) The Heaviside step function \( \Theta : \mathbb{R} \to \mathbb{R} \) is defined by
\[ \Theta(x) = \begin{cases} 
    0, & x < 0 \\
    1, & x \geq 0 
\end{cases} \]  \hfill (2)

Show that \( \Theta' = \delta \).
(f) Calculate
\[ \frac{1}{2\pi} \int_{\mathbb{R}} dy \exp \left( -\frac{\sigma^2 y^2}{2} \pm ixy \right) \]  
for both signs and prove the integral representation
\[ \delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} dy \exp(\pm ixy). \]  

Using Eq. (4), determine the Fourier transform \( \bar{\delta}(k) \) of \( \delta(x) \) and compare it with the Fourier transform of \( \delta(x) \) that is obtained directly based on the definition of the Fourier transform (see ex. 6, sheet 2).

8. Quantum mechanics vs. diffusion
In the lecture it has been shown that a wave packet in a one-dimensional space, which describes a free quantum particle of mass \( m \), spreads. During this process, the width
\[ b_{QM}(t) = b_{QM}(0) \sqrt{1 + \left(\frac{t}{\tau}\right)^2} \]
increases with time, where \( \tau = \frac{mb_{QM}(0)^2}{2\hbar} \) is the characteristic time scale. The constant \( b_{QM}(0) \) is related to the initial probability distribution of the wave packet via
\[ |\Psi(x, t = 0)|^2 = \frac{1}{\sqrt{2\pi(b_{QM}(0)/2)^2}} \exp \left( -\frac{x^2}{2(b_{QM}(0)/2)^2} \right). \]

In contrast, the width \( b_D(t) \) of the spatial probability density of a classical particle which diffuses in a one-dimensional space grows as \( b_D(t) = \sqrt{2Dt}, \ t \gg t_{\text{free}} \), where \( D \) is the diffusion constant and \( t_{\text{free}} \) is the mean time between two collisions of the particle with the surrounding particles.

(a) Sketch and discuss \( b_{QM}(t) \) and \( b_D(t) \).
(b) Calculate the time \( t^* \) for which \( b_{QM}(t^*) = b_D(t^*) \) and \( b_{QM}(t > t^*) > b_D(t > t^*) \) holds.
(c) Calculate \( t^* \) for
i. a \( N_2 \) molecule \( (b_{QM}(0) = 10^{-10} \text{ m}, \ m = 5 \cdot 10^{-26} \text{ kg}, \ D = 2 \cdot 10^{-5} \text{ m}^2\text{s}^{-1}) \), and
ii. a colloidal particle \( (b_{QM}(0) = 10^{-6} \text{ m}, \ m = 8 \cdot 10^{-15} \text{ kg}, \ D = 2 \cdot 10^{-13} \text{ m}^2\text{s}^{-1}) \).