16. Asymmetric potential well

Consider the asymmetric potential well

\[ U(x) = \begin{cases} 
U_1, & x < -a \\
0, & x \in [-a, a] \\
U_2, & x > a 
\end{cases} \]  

(1)

with \( a > 0 \) and \( 0 < U_1 \leq U_2 \) in one spatial dimension.

(a) Sketch \( U(x) \) as a function of \( x \) in a diagram.

(b) Make an appropriate ansatz for the eigenfunctions of the bound states (i.e., states with \( \varepsilon < U_1 \)) and formulate conditions to determine the coefficients.

(c) Derive an equation for the energy eigenvalues \( \varepsilon < U_1 \) and discuss the number of bound states as a function of \( U_1, U_2, \) and \( a \).

(d) [Bonus problem] Qualitatively discuss the solutions of the Schrödinger equation for \( \varepsilon > U_1 \).

17. Analytically solvable Schrödinger equation

Consider a quantum particle in one spatial dimension bound in the potential

\[ U(x) = -\frac{6}{\cosh^2 x}. \]  

(2)

(a) Sketch \( U(x) \) and determine a lower bound for the energy eigenvalues \( \varepsilon \).

(b) Formulate the time-independent Schrödinger equation for the eigenfunction \( \phi(x) \) corresponding to the eigenvalue \( \varepsilon \).

(c) Determine the symmetry properties of the eigenfunctions \( \phi(x) \).

(d) Consider the transformation between the variables \( x, \phi \) and \( y, \psi \) defined by \( \psi(y) = \phi(x) \cosh x \) and \( y = \sinh(x) \). By making use of the time-independent Schrödinger equation obtained in part (b), derive a differential equation for \( \psi(y) \).

(e) Show that \( \psi(y) \simeq Ay^k \) for \( |y| \gg 1 \) with constants \( A, k \). Determine an upper bound for \( k \) as well as the relationship between \( k \) and \( \varepsilon \).

(f) By using the results of the preceding parts of the problem, determine the possible energy eigenvalues \( \varepsilon < 0 \) and the associated eigenfunctions \( \phi(x) \).