18. **Commutators**

Consider the operators \( \hat{K}_1, \ldots, \hat{K}_6 \) defined in position space by

\[
\begin{align*}
\hat{K}_1 \psi(x) &= x^2 \psi(x), \\
\hat{K}_2 \psi(x) &= \left( i \frac{\partial}{\partial x} + a \right) \psi(x), \\
\hat{K}_3 \psi(x) &= x \frac{\partial \psi}{\partial x}(x), \\
\hat{K}_4 \psi(x) &= \psi(x)^*, \\
\hat{K}_5 \psi(x) &= (\psi(x))^2, \\
\hat{K}_6 \psi(x) &= \int_{-\infty}^{x} dy \, y \psi(y) 
\end{align*}
\]

with \( a \in \mathbb{C} \) and * denoting complex conjugation. Assume that the functions \( \psi : \mathbb{R} \rightarrow \mathbb{C} \) are differentiable at all orders and vanish sufficiently fast for \( |x| \rightarrow \infty \), such that \( \left| x^m \frac{\partial^n \psi(x)}{\partial x^n} \right| \) is bounded for all \( m, n \in \mathbb{N}_0 \).

(a) Which of the operators \( \hat{K}_1, \ldots, \hat{K}_6 \) are linear?

(b) Which of the operators \( \hat{K}_1, \ldots, \hat{K}_6 \) are Hermitian?

(c) Calculate the commutators \([\hat{K}_1, \hat{K}_2]\) and \([\hat{K}_3, \hat{K}_6]\).

19. **Operator identities**

Prove the following operator relations:

(a) \((\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger\), \((a\hat{A})^\dagger = a^* \hat{A}^\dagger\) with \( a \in \mathbb{C} \),

(b) \((\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}\), \((\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger\),

(c) \((\hat{A}^{-1})^{-1} = \hat{A}\), \((\hat{A}^\dagger)^\dagger = \hat{A}\), \((\hat{A}^{-1})^\dagger = (\hat{A}^\dagger)^{-1}\),

(d) \([\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}\).

20. **Hermitian operators**

Consider an operator \( \hat{A} \) and suppose that the relation \((\phi, \hat{A}\phi) = (\hat{A}\phi, \phi)\) applies to all states \( \phi \in \mathfrak{H} \). Show that this implies that \( \hat{A} \) is Hermitian, i.e., \((\psi, \hat{A}\phi) = (\hat{A}\psi, \phi)\) for any two states \( \phi, \psi \in \mathfrak{H} \).

(Hint: Choose two arbitrary \( \phi_1, \phi_2 \in \mathfrak{H} \) and apply the supposition to \( \phi = \phi_1 + \phi_2 \) and \( \phi = \phi_1 + i\phi_2 \).)