32. **Ehrenfest’s theorem in the Heisenberg picture**

Consider a quantum particle of mass $m$ in one-dimensional space governed by the (time-independent) Hamilton operator $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$, where $\hat{x}$ and $\hat{p}$ are the position and the momentum operator in the Schrödinger picture and the external potential $V$ is an analytic function.

(a) Let $\hat{A}$ and $\hat{B}$ be two operators fulfilling the commutator relation $[\hat{A}, \hat{B}] = 1$ and let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function. Show that $[f(\hat{A}), \hat{B}] = f'(\hat{A})$, where $f'$ is the derivative of $f$.

(Hint: Use ex. 27(a) of homework no. 11.)

(b) Calculate the commutators $[\hat{H}, \hat{x}(H)(t)]$ and $[\hat{H}, \hat{p}(H)(t)]$, where an operator $\hat{A}(H)(t) := \hat{U}(t)^\dagger \hat{A} \hat{U}(t)$ in the Heisenberg picture is related to the corresponding operator $\hat{A}$ in the Schrödinger picture via the time evolution operator $\hat{U}(t)$ (see lecture).

(c) Evaluate Heisenberg’s equations of motion $\frac{d}{dt} \hat{A}(H)(t) = i\hbar [\hat{H}, \hat{A}(H)(t)]$ for the position and momentum operator and, based on this, derive Ehrenfest’s theorem (see ex. 11 of homework no. 4).

33. **Expectation values in the Heisenberg picture**

Using the results of exercise 32, solve Heisenberg’s equations of motion for the position operator $\hat{x}(H)(t)$ and the momentum operator $\hat{p}(H)(t)$ for

(a) a free particle ($V(\hat{x}) = 0$), and

(b) a one-dimensional harmonic oscillator ($V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$).

Calculate and discuss the expectation values $\langle \hat{x} \rangle_t$ and $\langle (\hat{x} - \langle \hat{x} \rangle_t)^2 \rangle_t$ at time $t \geq 0$ for both cases (a) and (b), provided the quantum state at time $t = 0$ is given in position space by the wave function

$$\psi(x, t = 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{4\sigma^2} + ikx\right),$$

with $\sigma > 0$ and $k \in \mathbb{R}$.  

please turn over
34. **Minimal wave packet**

For two Hermitian operators \( \hat{A} \) and \( \hat{B} \) the Heisenberg uncertainty relation

\[
\Delta \hat{A} \Delta \hat{B} \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|,
\]

with \( \Delta \hat{C} := \sqrt{\langle (\hat{C} - \langle \hat{C} \rangle)^2 \rangle} \), has been proven in the lecture. A state \( |\psi\rangle \) is called a *minimal wave packet* if in Eq. (2) equality holds.

(a) Show that a state \( |\psi\rangle \) is a minimal wave packet if

\[
(\hat{A} - \langle \hat{A} \rangle)|\psi\rangle = ir(\hat{B} - \langle \hat{B} \rangle)|\psi\rangle \quad \text{or} \quad (\hat{B} - \langle \hat{B} \rangle)|\psi\rangle = ir(\hat{A} - \langle \hat{A} \rangle)|\psi\rangle
\]

with a real number \( r \in \mathbb{R} \).

(b) Consider spatial dimension \( d = 1 \). Formulate Eq. (3) for \( \hat{A} := \hat{x} \) and \( \hat{B} := \hat{p} \) in position space representation and solve the resulting differential equation for the wave function \( \psi(x, t) \) with the ansatz \( \psi(x, t) =: \exp(\phi(x, t)) \).

(c) Using the results of part (b), check whether the Gaussian wave packet of a free particle (see lecture)

\[
\psi(x, t) = \frac{1}{(2\pi \Lambda(t))^{1/4}} \exp\left(- \frac{(x - \frac{\hbar q t}{m})^2}{4\Lambda(t)} + i\left(qx - \frac{\hbar q^2}{2m} t\right)\right), \quad \Lambda(t) = \sigma^2 + i\frac{\hbar}{2m}t
\]

with \( \sigma > 0 \) and \( q \in \mathbb{R} \) is a minimal wave packet.

(d) Consider a one-dimensional harmonic oscillator of frequency \( \omega > 0 \) in the coherent state with parameter \( \gamma \in \mathbb{C} \) (see ex. 21 of homework no. 9). The corresponding wave function \( \psi_\gamma(x, t) \) fulfills \( \hat{b}\psi_\gamma(x, t) = \gamma \exp(-i\omega t)\psi_\gamma(x, t) \) with the annihilation operator \( \hat{b} = \sqrt{\frac{\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \). Check by means of the results of part (b) whether the coherent state with parameter \( \gamma \in \mathbb{C} \) is a minimal wave packet.