Guaranteed $\mathcal{H}_2$ Performance in Distributed Event-based State Estimation

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***Abstract*—Multiple agents sporadically exchange data over a broadcast network according to an event-based protocol to observe and control a dynamic process. The synthesis problem of each agent's state estimator and event generator, which decides whether information is broadcast or not, is addressed in this paper. In particular, a previously proposed LMI-synthesis procedure guaranteeing closed-loop stability is extended to incorporate an $\mathcal{H}_2$ performance measure. The improved estimation performance of the extended design is illustrated in simulations of an inverted pendulum, which is stabilized by two agents.**

I. Introduction

Present day control systems are mostly implemented on digital hardware, where the algorithms for sensing, estimation and control are typically run at a constant, predefined rate. This has the drawback that computation and communication resources are used at predetermined time instants, irrespective of the current state of the system or the information content of the measured data. In contrast, in event-based communication, estimation, and control, information is shared only when needed. This leads to a reduction in communication, ideally with only small performance losses. An overview on event-based control and estimation can be found in [1]–[4].

Event-based strategies are therefore especially promising for systems where communication represents a bottleneck. This is the case for systems analyzed herein, where multiple sensor-actuator agents observe and control a dynamic system and exchange information over a common communication medium, as shown in Fig. 1. This set-up was proposed and successfully demonstrated in experiments on an inverted pendulum system in [5]. The fundamental problem addressed in the following is how to design each agent's state estimator and event generator, such that a given estimation performance is met.

In [6], a procedure for the synthesis of stabilizing observer gains is presented. In contrast to earlier work, [5], these stability guarantees include the case of differences between any two agents' estimates, which stem, for example, from imperfect communication. While the focus in [6] is entirely on closed-loop stability, no performance criterion is included in the LMI-design, and the event generators are considered to be fixed. Herein, we augment the design of [6] by incorporating an $\mathcal{H}_2$ performance criterion, and by addressing both, the design of the state estimator and the event generator. The proposed LMI-based synthesis for event-based estimation is flexible: emphasis in the design can be placed on estimation performance or on the reduction of communication, for example.

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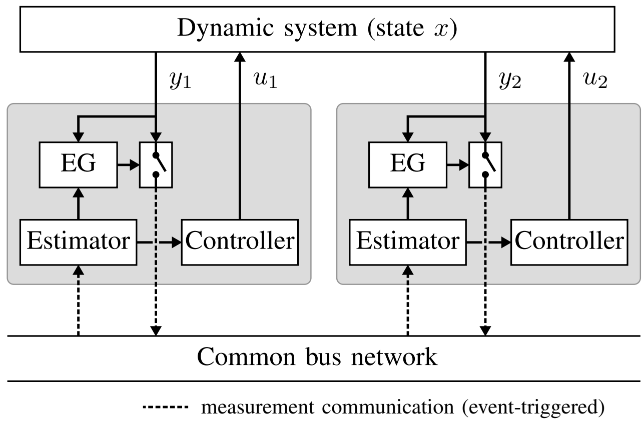

Figure 1. Block diagram of the distributed event-based estimation and control architecture considered in this paper (here with two agents). Each agent has access to measurements y_i and can thus observe a part of the system's state x. Based on the agent's state estimate and the local measurement y_i the event generator (EG) decides whether the measurement is communicated over the common bus network. The controller uses the local state estimate to compute the input u_i for the local actuator.

Related work: In [5], [7] the event generator is designed such that a centralized control and observer design can be applied to the distributed architecture. The main advantage of this approach is its simplicity since well-known centralized control design tools can be used to design the control and observer gains. However, in contrast to the synthesis procedure presented herein, the control design in [5], [7] does not explicitly account for the distributed structure of the control system.

Designs for distributed event-based estimation have been formulated as LMI-optimization in [8]–[10]. The references consider communication between agents according to a graph structure, and employ simpler event triggering mechanisms than the ones used herein. Triggering is based on the difference of the current measurement (or state) to the last transmitted one, while we exploit model-based predictions and consider the difference of the measurement to its prediction (rather than the last value sent). Moreover, the references exclusively treat the estimation problem, while we simultaneously address stability and performance of the distributed event-based control system that results when local estimates are used for feedback control.

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Outline of the paper: The distributed event-based estimation framework and the LMI-design from [6] are summarized in Sec. II. The augmented synthesis procedure, accounting for an $\mathcal{H}_2$ performance criterion is discussed in Sec. III. It is evaluated subsequently on a simulation example in Sec. IV, where different designs are compared, and the trade-off between communication and performance is discussed. The paper concludes with remarks in Sec. V.

II. Problem Formulation

The problem setting is similar to [5]–[7]. We consider the discrete-time linear system

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1) \quad (1)$$
$$y(k) = Cx(k) + w(k) \quad (2)$$

with time index k, state $x(k) \in \mathbb{R}^n$, control input $u(k) \in \mathbb{R}^q$, measurement $y(k) \in \mathbb{R}^p$, disturbances $v(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^p$, and all matrices of corresponding dimensions; (A, B) and (A, C) are assumed to be stabilizable and detectable. We assume that there are N sensor-actuator agents, each of them measuring a portion of the input and output. Decomposing the input vector u and output vector y accordingly leads to

$$B\,u(k-1) = [B_1 \;\; B_2 \;\; \ldots \;\; B_N] \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_N(k) \end{bmatrix} \quad (3)$$

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_N(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} x(k) + \begin{bmatrix} w_1(k) \\ w_2(k) \\ \vdots \\ w_N(k) \end{bmatrix}, \quad (4)$$

where $u_i(k) \in \mathbb{R}^{q_i}$ is agent i's input and $y_i(k) \in \mathbb{R}^{p_i}$ its measurement. Note that local stabilizability or detectability is *not* required, hence (A, B_i) may be not stabilizable and (A, C_i) may be not detectable. We assume further that the disturbances $v(k)$ and $w_i(k)$, $i = 1, 2, \ldots, N$, are zero mean, independent and identically distributed for all k and have covariances V and W_i respectively. For guaranteeing input-to-state stability in the sense of [6], $v(k)$ and $w_i(k)$ need additionally to be bounded, hence they could be for example uniformly distributed or be modeled by truncated Gaussian distributions.

A. Distributed State Estimation and Communication

The agents communicate sensor data $y_i(k)$ over a broadcast network. If one agent communicates, all other agents will receive the data. Unlike [5] and [7], the agents do not share their input data $u_i(k)$ with each other. Agents are assumed to be synchronized in time, and network communication is assumed to be instantaneous and without delay.

The following rule is used for deciding whether agent i should broadcast its local measurement $y_i(k)$ or not:

$$\text{transmit } y_i(k) \;\Leftrightarrow\; ||\Delta_i^{-1}(y_i(k) - C_i\hat{x}_i(k|k-1))||_2 \geq 1, \quad (5)$$

where $\Delta_i \in \mathbb{R}^{p_i \times p_i}$ is symmetric, positive definite ($\Delta_i > 0$), $\hat{x}_i(k|k-1)$ is agent i's belief of the state $x(k)$ based on measurements until time $k-1$ (which is made precise below), $||\cdot||_2$ denotes the vector 2-norm, and $C_i\hat{x}_i(k|k-1)$ is agent i's prediction of its measurement $y_i(k)$. Note that the transmit rule used herein is a slight generalization of the one in [6], where Δ_i was restricted to be of the form $\delta_i I$, with $\delta_i \in \mathbb{R}$ and $I \in \mathbb{R}^{p_i \times p_i}$ the identity matrix. Nonetheless the results from [6] hold likewise for the more general transmit decision given by (5). The decision rule used herein can be motivated by the potentially non-homogeneous information content of the elements of y_i, as well as different physical units. In addition, the Δ_i will enter the design process as decision variables.

At each time instant k, the index set of measurements transmitted is denoted by

$$I(k) := \{i \mid 1 \leq i \leq N, ||\Delta_i^{-1}(y_i(k) - C_i\hat{x}_i(k|k-1))||_2 \geq 1\}. \quad (6)$$

Agent i's state estimate is given by the following recursive update rule:

$$\hat{x}_i(k|k-1) = A\hat{x}_i(k-1|k-1) + B\hat{u}^i(k-1) \quad (7)$$
$$\hat{x}_i(k|k) = \hat{x}_i(k|k-1) + \sum_{j \in I(k)} L_j\big(y_j(k) - C_j\hat{x}_i(k|k-1)\big) + d_i(k), \quad (8)$$

where $\hat{u}^i(k-1) \in \mathbb{R}^q$ denotes agent i's belief of the input vector $u(k-1)$, L_j are observer gains to be designed, and $d_i(k)$ represents a disturbance, which is assumed to be bounded. The disturbance d_i has been introduced in [7] to model mismatches between the estimates of the individual agents, which may originate from, for example, unequal initialization, different computation accuracy, or imperfect communication.

B. Distributed Control

It is assumed that a state-feedback controller $F \in \mathbb{R}^{q \times n}$ is given such that $A + BF$ is asymptotically stable (magnitude of all eigenvalues strictly less than one). The control $u_i(k)$ on agent i is computed as

$$u_i(k) = F_i\hat{x}_i(k) \quad (9)$$

where $F^{\mathrm{T}} = (F_1^{\mathrm{T}}, F_2^{\mathrm{T}}, \ldots, F_N^{\mathrm{T}})$ is the decomposition of the state feedback gain F. Each agent uses the estimate

$$\hat{u}^i(k) = F\hat{x}_i(k|k) \quad (10)$$

of $u(k)$ to update his state estimate according to (7).

C. Closed-Loop System

In [6], conditions guaranteeing the stability of the closed-loop dynamics, which result from (1), (2), (5), (7), (8), (9), and (10) were derived. These conditions are summarized with the following theorem:

Theorem 2.1: (From [6]) Let the matrix inequalities

$$A_{\mathrm{cl}}^{\mathrm{T}}(\Pi_i)PA_{\mathrm{cl}}(\Pi_i) - P < 0 \quad \text{and}$$
$$((I - LC)A)^{\mathrm{T}}\, Q(I - LC)A - Q < 0$$

with

$$A_{\mathrm{cl}}(\Pi_i) := (I - \sum_{m \in \Pi_i} L_m C_m)(A + BF) \quad \text{and}$$
$$L := (L_1, L_2, \ldots, L_N)^{\mathrm{T}}$$

be fulfilled for positive definite matrices $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$, $P > 0$, $Q > 0$, and for all permutations $\Pi_i \in \Pi$, where

Π is defined as the set of all permutations of $\{1, 2, \dots, N\}$.[1] Then, the closed-loop dynamics are input-to-state stable.

Input-to-state stability, as introduced in [11], implies that the state trajectory remains bounded provided that bounded inputs are applied to the system and can be viewed as an extension of Lyapunov stability to systems with inputs. If suitable matrices P and Q are found, Theorem 2.1 guarantees stability of the closed-loop system under all possible switching sequences, which are represented by the permutations Π. Therefore, the result is also independent of the communication thresholds Δ_i.

Using the restriction $P = Q$, which is needed to obtain a linear dependence on the decision variables (as will become clear in the following), and applying the Schur complement, [12, p.650], the following linear matrix inequalities (LMIs) ensuring closed loop stability are obtained:

$$\begin{pmatrix} P & PA_{\text{cl}}(\Pi_i) \\ A_{\text{cl}}(\Pi_i)^{\mathrm{T}}P & P \end{pmatrix} > 0 \tag{11}$$

for all permutations $\Pi_i \in \Pi$, and

$$\begin{pmatrix} P & P(I - LC)A \\ A^{\mathrm{T}}(I - LC)^{\mathrm{T}}P & P \end{pmatrix} > 0. \tag{12}$$

The LMIs (11), (12) are used in [6] to design stabilizing observer gains L_i. Whether the resulting semidefinite program is feasible, depends on the problem parameters. In this paper, the LMI design is augmented to express performance criteria in addition to the closed-loop stability requirement. To simplify the following discussion, we make the following assumption:

Assumption 1: There exists L and P such that (11) and (12) are satisfied.

If this assumption is violated for a specific problem, a modified procedure was introduced in [6], which uses a reset strategy for the agents' estimates to still guarantee input-to-state stability at the price of additional inter-agent communication. The derivations presented in the remainder can also be used to augment the LMI design in case Assumption 1 is not satisfied, since the modified communication protocol from [6] can be applied accordingly.

D. Objective

The objective of this paper is to design the observer gains L_i and communication thresholds Δ_i, such that the resulting closed-loop system, given by (1), (2), (5), (7), (8), (9), (10), is input-to-state stable for bounded disturbances v, w_i, d_i, and such that a certain $\mathcal{H}_2$ performance is achieved. While stability was extensively discussed in [6], this article focuses on the aspects related to the closed-loop performance. We propose to divide the synthesis procedure into two steps. By considering the full communication case in the first step, a lower bound on the achievable $\mathcal{H}_2$ cost can be minimized, yielding the observer gains L_i. In a second step, the communication thresholds Δ_i are chosen such that the given performance requirements are met. By doing so, communication can be systematically traded off for performance. In contrast to the joint optimization of the L_i's and Δ_i's, the two step procedure yields convex problems, thus allowing for an efficient solution.

[1] A permutation is defined as the drawing of zero up to N elements from $\{1, 2, \dots, N\}$ without repetition and without considering the order. Therefore Π contains the empty set and has cardinality 2^N.

III. Augmented LMI-Synthesis

In this section, we derive a general $\mathcal{H}_2$ performance measure, which can be used, for example, to express the objectives of minimizing the estimation error or minimizing communication rates. The performance measure will be used to augment the LMIs (11) and (12), responsible for ensuring stability of the distributed event-based control system.

To that extent, we first reformulate the agent error dynamics in Sec. III-A to simplify the subsequent analysis. We introduce the general $\mathcal{H}_2$ performance measure in Sec. III-B and highlight two particular instances, which allow to reduce the estimation error or the average communication. In Sec. III-C, the synthesis procedure guaranteeing a worst case $\mathcal{H}_2$ performance is presented.

A. Agent Error Dynamics

The time evolution of agent i's estimation error is given by the combination of (1), (7), (8), (9), and (10),

$$\begin{aligned} e_i(k) &= Ae_i(k-1) + \sum_{j=1}^{N} B_j F_j \epsilon_{ji}(k-1) \\ &- \sum_{j \in I(k)} L_j(y_j(k) - C_j\hat{x}_i(k|k-1)) - d_i(k) + v(k-1), \end{aligned}$$

where the agent error (of agent i) is defined as $e_i(k) := x(k) - \hat{x}_i(k|k)$ and the inter-agent error (between agent i and j) as $\epsilon_{ji}(k) := \hat{x}_j(k|k) - \hat{x}_i(k|k)$. Hence, in the absence of disturbances, $d_i(k) = 0$, and assuming zero inter-agent error at time $k = 0$, it follows that $\hat{x}_i(k|k-1) = \hat{x}_j(k|k-1)$ and $e_i(k) = e_j(k)$ for all $k > 0$.

For deriving the performance criterion, we consider the simplified dynamics without disturbances d_i and with identical estimates in the interest of a tractable design. The proposed performance measure can be extended to incorporate nonzero disturbances d_i and a nonzero inter-agent error, e.g. by using worst-case upper bounds as provided in [6, proof of Lemma 3.1]. By augmenting the final optimization problems with the conditions (11) and (12), closed-loop stability can be guaranteed even when the agents' state estimates differ and the disturbances d_i are nonzero. Under these assumptions, the agent error dynamics can be rewritten as (see [6, equation (15)] with $d_i(k) = 0, \hat{x}_i(k|k) = \hat{x}_j(k|k), \forall i, j, k$)

$$\begin{aligned} e_i(k) = (I - LC)Ae_i(k-1) + (I - LC)v(k-1) - \\ \sum_{j=1}^{N} L_j w_j(k) + \xi(k), \end{aligned} \tag{13}$$

where $\xi(k)$ is given by

$$\xi(k) = \sum_{j \in \bar{I}(k)} L_j(y_j(k) - C_j\hat{x}_j(k|k-1)). \tag{14}$$

The index set of measurements which are not transmitted is denoted by $\bar{I}(k)$, hence

$$\bar{I}(k) := \{i \mid 1 \le i \le N, ||\Delta_i^{-1}(y_i(k) - C_i\hat{x}_i(k|k-1))||_2 < 1\}. \tag{15}$$

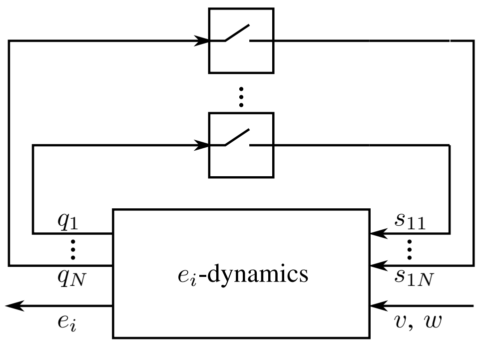

Figure 2. Block diagram of the agent error dynamics. The error e_i is driven by the external disturbances v and w. The switches and the signals q_i and s_{1i} are used to model the event-based communication. Based on the magnitude of the signal $q_i(k)$ at time instant k, the i'th switch is either closed (no communication in case $||q_i(k)||_2 < 1$) implying $q_i(k) = s_{1i}(k)$ or opened (communication in case $||q_i(k)||_2 \geq 1$) implying $s_{1i}(k) := 0$.

The communication protocol guarantees by construction that $||\Delta_j^{-1}(y_j(k) - C_j\hat{x}_j(k|k-1))||_2$ is less than one for all $j \in \bar{I}(k)$, and therefore $\xi(k)$ can be rewritten as

$$\xi(k) = \sum_{j=1}^{N} L_j \Delta_j s_{1j}(k), \tag{16}$$

with $||s_{1j}(k)||_2 < 1$ and $s_{1j}(k) := 0$ if $j \notin \bar{I}(k)$. In other words, $s_{1j}(k)$ is given by

$$s_{1j}(k) := \mathbf{1}_{j \in \bar{I}(k)}\; q_j(k), \tag{17}$$
$$q_j(k) : = \Delta_j^{-1}(y_j(k) - C_j\hat{x}_j(k|k-1)), \tag{18}$$

where $\mathbf{1}_{j \in \bar{I}(k)}$ denotes the indicator function; that is, the function is 1 if and only if the statement in the subscript is true and zero otherwise. The agent-error dynamics can therefore be illustrated by the block diagram as shown in Fig. 2: in case that there is no communication of a specific measurement y_j, the j'th switch is closed and the error signal q_j is fed back. In case that the communication is triggered, the j'th switch remains open, resulting in $s_{1j}(k) = 0$.

By normalizing the exogenous inputs v and w,

$$s_{2j} := W_j^{-\frac{1}{2}} w_j, \quad s_3 := V^{-\frac{1}{2}} v, \tag{19}$$
$$s_1 := (s_{11}, \dots, s_{1N})^{\mathrm{T}}, \quad s_2 := (s_{21}, \dots, s_{2N})^{\mathrm{T}} \tag{20}$$

the agent error dynamics (13) can be rewritten as

$$e_i(k) = \hat{A}e_i(k-1) + \hat{B}_1 s_1(k) + \hat{B}_2 s_2(k) + \hat{B}_3 s_3(k-1), \tag{21}$$

with

$$\hat{A} = (I - LC)A, \qquad \hat{B}_1 = (L_1\Delta_1, \dots, L_N\Delta_N)$$
$$\hat{B}_2 = (L_1 W_1^{\frac{1}{2}}, \dots, L_N W_N^{\frac{1}{2}}), \quad \hat{B}_3 = (I - LC)V^{\frac{1}{2}}.$$

B. $\mathcal{H}_2$ Performance Measure

We choose

$$z(k) = \hat{C}e_i(k-1) + \hat{D}_2 s_2(k) + \hat{D}_3 s_3(k-1), \tag{22}$$

$k = 1, 2, \dots,$ as virtual output and use the power semi-norm,

$$J = \lim_{n\to\infty} \sqrt{\frac{1}{n}\sum_{k=1}^{n} z(k)^{\mathrm{T}} z(k)} =: ||z||_{\mathcal{P}} \tag{23}$$

as performance measure, [13, p.103] (provides a continuous time definition; the discrete-time case is analogous). Note that in discrete time, the $\mathcal{H}_2$ norm is well-defined even for systems with direct feed-through.

In case e_i was stationary and ergodic, (23) would be equivalent to the expected standard deviation of the output z, [14, p.224]. However, due to the communication protocol and resulting nonlinear feedback (see Fig. 2), e_i is not guaranteed to be stationary. The cost (23) is well-defined, since the agent error remains bounded for bounded disturbances v, w_j. In addition, (22) offers flexibility in the choice of $\hat{C}$, $\hat{D}_2$, and $\hat{D}_3$. Two particular choices are highlighted next:
1) Choosing $\hat{C} = I$, $\hat{D}_2 = 0$, $\hat{D}_3 = 0$ uses directly the power of the agent estimation error as performance criterion and corresponds to a steady-state Kalman filter design.
2) Note that $q_j(k)$ given by equation (18) can be rewritten as

$$q_j(k) = \Delta_j^{-1}\Big(C_j A e_j(k-1) + W_j^{\frac{1}{2}} s_{2j}(k) + C_j V^{\frac{1}{2}} s_3(k-1)\Big), \tag{24}$$

since all agents' state estimates are assumed to be equal for establishing the performance measure. Hence, by choosing

$$\hat{C}^{\mathrm{T}} = ((\Delta_1^{-1}C_1A)^{\mathrm{T}}, \dots, (\Delta_N^{-1}C_NA)^{\mathrm{T}}), \tag{25}$$
$$\hat{D}_2 = \mathrm{diag}\left(\Delta_1^{-1}W_1^{\frac{1}{2}}, \dots, \Delta_N^{-1}W_N^{\frac{1}{2}}\right), \quad \text{and} \tag{26}$$
$$\hat{D}_3^{\mathrm{T}} = ((\Delta_1^{-1}C_1V^{\frac{1}{2}})^{\mathrm{T}}, \dots, (\Delta_N^{-1}C_NV^{\frac{1}{2}})^{\mathrm{T}}) \tag{27}$$

we have that $z = q$ enabling the direct minimization of the power of q. The signal q is directly related to the communication, since a communication is triggered if $||q_j||_2 \geq 1$.

Thus, design 1) seeks to optimize estimation performance, whereas design 2) aims at reducing communication.

C. Synthesis

Minimizing (23) directly is difficult, due to the nonlinear feedback of the q_j's (see Fig. 2), leading in particular to a correlation of the distrubances v and w with the s_{1j}'s. Moreover, the estimation error e_i is not stationary, which makes it difficult to use a probabilistic characterization of $||z||_{\mathcal{P}}$.

Therefore, the following alternative approach is proposed: We minimize first (23) with respect to the observer gains L_i for the full communication case, where all agents communicate at every step (i.e. $s_1 = 0$). Note that if the communication is reduced, the agents have less information leading to a potential performance degradation. Hence, the full communication scenario yields a lower bound on the achievable cost (23). In a second step, the fact that $||s_{1j}||_2 < 1$ for all j's is exploited, enabling the derivation of an upper bound on the cost (23). Thus, the communication thresholds Δ_i are designed to reduce the communication, while at the same time guaranteeing a worst case performance in terms of (23). The estimator gains

L_i are fixed during the synthesis of the Δ_i in order to obtain a convex problem.

An upper bound to (23) can be obtained by applying the triangle inequality (recall that $||\cdot||_{\mathcal{P}}$ is a semi-norm):

$$J \leq ||g_1 * s_1||_{\mathcal{P}} + ||g_2 * s_2 + g_3 * s_3||_{\mathcal{P}}, \tag{28}$$

where $*$ denotes the convolution operator, g_1, g_2, and g_3 the impulse responses from s_1, s_2, and s_3 to z. The first term can be upper bounded by

$$||g_1 * s_1||_{\mathcal{P}} \leq ||G_1||_\infty ||s_1||_{\mathcal{P}} \leq \sqrt{N}||G_1||_\infty, \tag{29}$$

where G_1 is the Z-transform of g_1, and $||G_1||_\infty$ denotes the $\mathcal{H}_\infty$ norm of G_1, see e.g. [13, p.107] (provides a continuous time derivation; the discrete-time case is analogous). Note that $||s_1||_{\mathcal{P}} \leq \sqrt{N}$ since all the s_{1j}'s have magnitude less than one.

The inputs s_2 and s_3 are independent, identically distributed, and have unit variance; and therefore $||g_2 * s_2 + g_3 * s_3||_{\mathcal{P}}$ can be expressed by

$$||g_2 * s_2 + g_3 * s_3||_{\mathcal{P}} = ||G_2||_2 + ||G_3||_2, \tag{30}$$

where G_2, G_3 denote the Z-transforms of g_2 and g_3, respectively and $||G||_2$ the $\mathcal{H}_2$ norm of the transfer function G, [13, p.107]. Hence, the following upper bound on the performance measure J is obtained,

$$J \leq \sqrt{N}||G_1||_\infty + ||G_2||_2 + ||G_3||_2. \tag{31}$$

For the limit $\Delta_j \to 0$, the full communication scenario is recovered, since in that case $||G_1||_\infty$ vanishes (note the linear dependence of $\hat{B}_1$ on the Δ_i), and $J = ||G_2||_2 + ||G_3||_2$.

Summarizing, the following procedure for designing the observer gains L_i together with the communication thresholds Δ_i, $i = 1, \ldots, N$, is proposed:
1) Design the observer gains L_i, which minimize $||G_2||_2 + ||G_3||_2$, subject to the constraints (11) and (12) ensuring closed-loop stability. The communication thresholds Δ_i are kept fixed.
2) Choose the communication thresholds Δ_i as large as possible, i.e. maximize for instance the trace of the Δ_i's, but such that $\sqrt{N}||G_1||_\infty + ||G_2||_2 + ||G_3||_2 < J_{\max}$, where $J_{\max}$ is a predefined upper bound on the closed-loop performance. Keep the observer gains L_i fixed.

Step 1) represents a nominal control design, based on the full communication scenario. The $\mathcal{H}_2$ norm $||G_2||_2 + ||G_3||_2$ is a lower bound on J. Step 2) provides a means for choosing the communication thresholds Δ_i, such that the desired performance $J_{\max}$ is guaranteed. Clearly, the desired performance $J_{\max}$ must be larger than $||G_2||_2 + ||G_3||_2$.

Both optimization problems can be formulated as semidefinite programs and are therefore efficiently solvable using standard software packages. In particular, it holds that $(||G_2||_2 + ||G_3||_2)^2 < \mathrm{tr}(H)$, where $H \in \mathbb{R}^{(n+p)\times(n+p)}$ is symmetric positive definite, if and only if there exists a positive definite matrix P such that

$$\begin{pmatrix} I & 0 & \hat{C} \\ 0 & P & P\hat{A} \\ \hat{C}^{\mathrm{T}} & \hat{A}^{\mathrm{T}}P & P \end{pmatrix} > 0, \begin{pmatrix} I & 0 & \hat{D}_{23} \\ 0 & P & P\hat{B}_{23} \\ \hat{D}_{23}^{\mathrm{T}} & \hat{B}_{23}^{\mathrm{T}}P & H \end{pmatrix} > 0, \tag{32}$$

with $\hat{B}_{23} = (\hat{B}_2, \hat{B}_3)$ and $\hat{D}_{23} = (\hat{D}_2, \hat{D}_3)$, see Appendix. Note that $\mathrm{tr}(H)$ refers to the trace of the matrix H. By adding equation (32) to the conditions (11) and (12), the following semidefinite program is obtained for step 1):

$$\min \mathrm{tr}(H) \quad \text{subject to} \quad (11), (12), \text{and } (32). \tag{33}$$

Note that the linear matrix inequalities (11), (12), and (32) are linear in the optimization variables P, $Y_i := PL_i$, and H, provided that the Δ_i's are fixed.

Similarly, for the $\mathcal{H}_\infty$ norm of G_1, it holds that $||G_1||_\infty^2 < \gamma_1$ if and only if there exists a positive definite matrix P such that

$$\begin{pmatrix} P & \hat{A}P & \hat{B}_1 & 0 \\ P\hat{A}^{\mathrm{T}} & P & 0 & P\hat{C}^{\mathrm{T}} \\ \hat{B}_1^{\mathrm{T}} & 0 & I & 0 \\ 0 & \hat{C}P & 0 & \gamma_1 I \end{pmatrix} > 0, \tag{34}$$

see e.g. [15]. Equation (34) is linear in the Δ_i's (through $\hat{B}_1$), which allows to formulate the optimization problem in step 2) as

$$\begin{gathered} \max \sum_{i=1}^{N} \mathrm{tr}(\Delta_i) \quad \text{subject to} \\ (34) \quad \text{and} \quad \gamma_1 < \frac{1}{N}(J_{\max} - ||G_2||_2 - ||G_3||_2)^2. \end{gathered} \tag{35}$$

Note that, when solving (35), $||G_2||_2$ as well as $||G_3||_2$ are kept fixed. The motivation for maximizing the trace of the Δ_i is given by the observation that the q_i, which trigger the communication, are directly proportional to the inverse of the Δ_i, see (24). Alternatively one could maximize the minimum eigenvalues of the Δ_i, which would likewise result in a semidefinite program, or introduce a weighting in the objective function of the optimization problem (35). A weighting of the different communication thresholds could be interesting for applications where the communication costs differ among the agents.

Feasibility of (33) *and* (35)
Note that the first condition of (32) is equivalent to $\hat{A}^{\mathrm{T}}P\hat{A} - P + \hat{C}^{\mathrm{T}}\hat{C} < 0$ (Schur complement, [12, p.650]). Condition (12) implies that $\hat{A}^{\mathrm{T}}P\hat{A} - P < 0$ and therefore, by scaling P, the first inequality of (32) can always be satisfied, given that Assumption 1 holds. The second inequality of (32) is equivalent to $\hat{B}_{23}^{\mathrm{T}}P\hat{B}_{23} + \hat{D}_{23}^{\mathrm{T}}\hat{D}_{23} - H < 0$, which is likewise fulfilled by choosing the minimum eigenvalue of H large enough. This implies that under Assumption 1, (33) is always solvable.

The solvability of (35) follows trivially by noting the communication thresholds Δ_i can be chosen to be zero.

IV. Simulation Example

In this section, we present a simulation example to illustrate the proposed optimization framework for designing the event-based state estimators, namely the estimator gains L_i, and the triggering thresholds Δ_i. For the simulations, we use the same inverted pendulum example as in [6] and [7]. It is based on the Balancing Cube [16], which was used as the physical test bed for the experiments in [5].

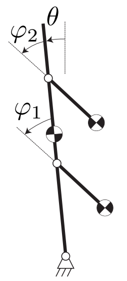

Figure 3. Inverted pendulum model: The pendulum is stabilized by the relative motion of its two arms.

A. Model

Consider the inverted pendulum system depicted in Fig. 3, where φ_1 and φ_2 parametrize the inclination of the first and second arm, and θ the inclination of the inverted pendulum. Choosing $x^{\mathrm{T}} = (\theta, \dot{\theta}, \varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$, the dynamics are given by

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1) \tag{36}$$

with

$$A = \begin{pmatrix} 1.0007 & 0.0100 & -0.0001 & -0.0005 & -0.0001 & -0.0012 \\ 0.1492 & 1.0007 & -0.0151 & -0.0462 & -0.0151 & -0.1231 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0.0005 & 0.0012 \\ 0.0461 & 0.1230 \\ 0.0100 & 0 \\ 1 & 0 \\ 0 & 0.0100 \\ 0 & 1 \end{pmatrix}.$$

The input vector is divided according to $u = (u_1, u_2)^{\mathrm{T}}$, with u_1 the desired angular velocity of the upper arm $u_1 = \dot{\varphi}_{1_{\text{des}}}$, and u_2 the desired angular velocity of the lower arm $u_2 = \dot{\varphi}_{2_{\text{des}}}$, see [6] and references therein for the modeling. The process noise is assumed to originate mainly from the uncertainties in the actuation and is therefore chosen as

$$v(k-1) = B\ (n_{u1}(k-1), n_{u2}(k-1))^{\mathrm{T}}, \tag{37}$$

where n_{u1} and n_{u2} are independent stochastic processes, which are uniformly distributed over $[-3.0\,^\circ/\mathrm{s}, 3.0\,^\circ/\mathrm{s}]$.

The lower control unit in Fig. 3 is called *agent* 1, and the upper one *agent* 2. Agent 1 has access to the noisy measurements $\varphi_1 + n_{\varphi_1}$, $\dot{\varphi}_1 + n_{\dot{\varphi}_1}$, and $\dot{\theta} + n_{\dot{\theta}}$ and computes u_1. Agent 2 has access to the noisy measurements $\varphi_2 + n_{\varphi_2}$, and $\dot{\varphi}_2 + n_{\dot{\varphi}_2}$. The noise signals $n_{\varphi_1}, n_{\dot{\varphi}_1}, n_{\dot{\theta}}, n_{\varphi_2}$, and $n_{\dot{\varphi}_2}$ are assumed to be independent, uniformly distributed with zero mean and variances $\sigma^2_{\varphi_i} = (0.05\,^\circ)^2$, $\sigma^2_{\dot{\varphi}_i} = (0.1\,^\circ/\mathrm{s})^2$, and $\sigma^2_{\dot{\theta}} = (0.24\,^\circ/\mathrm{s})^2$, $i = 1, 2$.

The system is controllable and observable, but neither controllable nor observable for each agent on its own. A stabilizing state feedback controller F is found via an LQ regulator approach, yielding

$$F = \begin{pmatrix} 212.5872 & 55.0168 & -19.3450 & -2.5374 & -23.5728 & -6.7664 \\ -84.9883 & -22.0881 & 6.4894 & 1.0187 & 6.3579 & 2.7166 \end{pmatrix}.$$

B. Design 1: $z(k) = e_i(k)$

The designs 1) and 2) as introduced in Sec. III-B are compared on the simulation example. First, we choose $z(k) = e_i(k)$, in analogy to a steady-state Kalman filter. Note however, that (11) and (12) are imposed to account for the distributed architecture of the control system. The obtained observer design is therefore fundamentally different from a centralized steady-state Kalman filter.

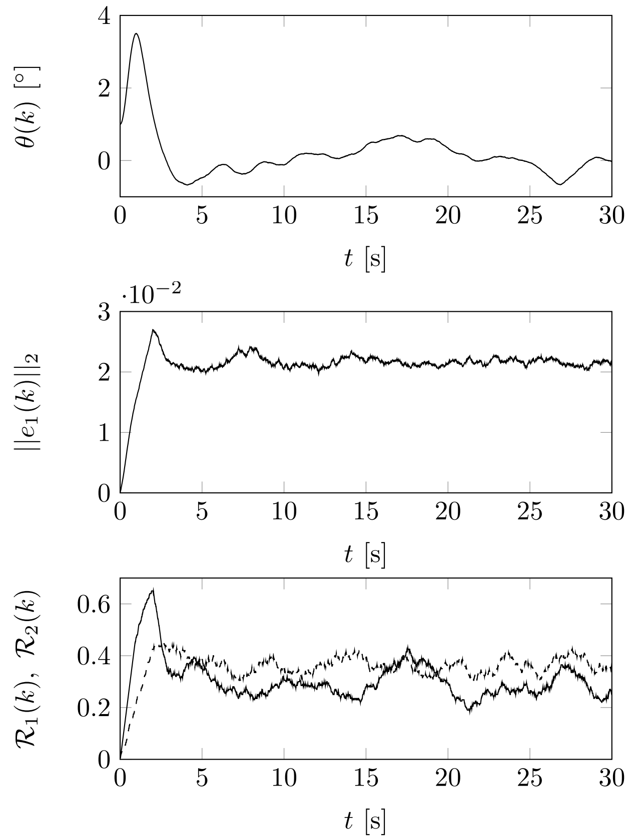

Figure 4. Top: time history of pendulum inclination angle $\theta(k)$ using a sampling time of $1\,\mathrm{ms}$. Middle: 2-norm of agent 1's estimation error (after smoothing with a 200-sample moving average filter). Bottom: estimation of the communication $\mathcal{R}_1$ of agent 1 (solid) and $\mathcal{R}_2$ of agent 2 (dashed) via a 200-sample moving average filter.

Due to the particular choice of $z(k) = e_i(k)$, the optimization problem (33) is independent of the communication thresholds Δ_i. Thus, (33) is solved first yielding the observer gains L_i, as well as the $\mathcal{H}_2$ gain $||G_2||_2 + ||G_3||_2$. In a second step, (35) is solved to design the communication thresholds Δ_i, while keeping the L_i's fixed.

Solving (33) yields $||G_2||_2 + ||G_3||_2 = 3.96 \cdot 10^{-3}$. To reduce communication, a comparably high maximum cost of $J_{\text{max}} = 0.1$ is tolerated. The resulting distributed state estimation is evaluated in simulations of the inverted pendulum model.

Disturbance Rejection Properties

In a first experiment, the simulation is started with initial conditions $x(0) = (1\,^\circ, 0, 0.1\,^\circ, 0, -0.1\,^\circ, 0)^{\mathrm{T}}$, $\hat{x}_1(0) = (0, 0, 0.1\,^\circ, 0, 0, 0)^{\mathrm{T}}$, and $\hat{x}_2(0) = (0, 0, 0, 0, -0.1\,^\circ, 0)^{\mathrm{T}}$. Hence, each agent knows its own inclination angle, but not the pendulum inclination angle. A packet drop is assumed to occur with a probability of 2%, i.e. every 50th measurement is lost on average. The resulting time histories of the pendulum inclination angle, the estimation error of agent 1, and the communication are depicted in Fig. 4. The communication rates $\mathcal{R}_1$ and $\mathcal{R}_2$ are normalized such that 1 corresponds to an agent communicating at every time instant.

Table I. COMMUNICATION RATES AND RESULTING $\mathcal{H}_2$ COST.

	$\mathcal{R}$	$\|\|e_i\|\|_\mathcal{P}$	$\|\|q\|\|_\mathcal{P}$
Design 1)	32.91%	0.0265	1.256
Design 2)	7.33%	0.0516	0.694

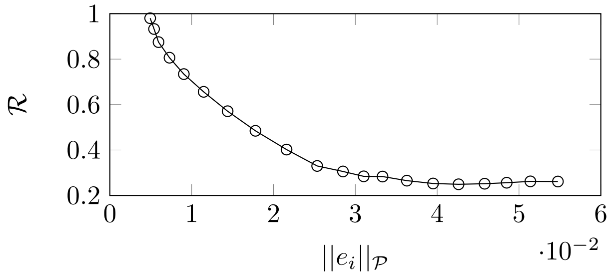

Figure 5. Total communication versus estimation performance. The total communication $\mathcal{R}$ is normalized such that 1 corresponds to both agents communicating at every time instant.

Steady-State Performance

In a second experiment, the steady state communication rate and the performance $||z||_\mathcal{P}$ is estimated. To that extent, the pendulum is initialized in upright position at rest, and the communication and performance is evaluated using averaging over $150\,\mathrm{s}$.[2] The results can be found in Tab. I. Note that the obtained performance is well below the bound $J_{\max}$, which is mainly due to the conservative worst case bound given by equation (29), since in practice $||s_1||_\mathcal{P} \ll \sqrt{N}$.

Influence of J_{max}

Additionally, the influence of $J_{\max}$ is studied by gradually increasing its value from $1.0 \cdot 10^{-2}$ to $2.0 \cdot 10^{-1}$. The resulting steady-state communication rate and performance $||z||_\mathcal{P}$ is shown in Fig. 5. Initially, a rapid decrease in communication, together with slight performance loss can be observed. As $J_{\max}$ is increased further, $||z||_\mathcal{P}$ increases linearly, whereas the communication rate flattens out.

C. Design 2: $z(k) = q(k)$

As an alternative design, we choose $z(k) = q(k)$ to penalize the communication directly. In this case, the optimization problem (33) is dependent on both, the L_i and the Δ_i, see (24). We therefore propose to solve the optimization problems (33) and (35) in an alternating manner, always keeping either the L_i or the Δ_i fixed. Initially, $\Delta_1 = \Delta_2 = 0.01I$ is chosen. The optimization problem (33) is solved to obtain the L_i, while keeping the Δ_i fixed, yielding $||G_2||_2 + ||G_3||_2 = 0.465$. Then (35) is solved with $J_{\max} = 120$ to update the Δ_i, while keeping the L_i fixed, before repeating the procedure a second time.[3] To estimate the steady-state communication and performance $||z||_\mathcal{P}$, the simulated trajectories (with zero initial conditions) are again averaged over $150\,\mathrm{s}$. In comparison to design 1, the average communication is significantly lower at the expense of increased estimation error, see Tab. I.

[2]The results were found to change insignificantly (below 2%) when increasing the time horizon.

[3]Repeating the procedure another time did, however, not improve the design; thus, we stopped after two iterations.

V. CONCLUSION

This paper extends the framework presented in [6] by augmenting the design of the distributed event-based state estimators with a performance measure. The proposed approach is flexible and encompasses different objectives such as minimizing the estimation error or the communication. In addition to providing a synthesis procedure for the observer feedback gains, a systematic approach for the design of the communication thresholds is presented. The algorithms are evaluated using a simulation example. In particular, the closed-loop performance resulting from two different performance measures is illustrated, and the trade-off between performance and communication is highlighted.

It is shown that the communication can be drastically reduced by choosing $z = q$ and optimizing for the L_i and Δ_i in an alternating manner, see Sec. IV-C. Future work will aim at a better understanding of this alternating optimization procedure with respect to numerical stability and convergence.

APPENDIX

In the following, the conditions (32) are derived, which express the $\mathcal{H}_2$ norm objective as LMIs. The results are well-known in the literature, [15], [17], but the authors were not able to find a formulation identical to (32).

Expressing the $\mathcal{H}_2$ norm objective as LMIs

Consider the discrete-time system

$$\begin{aligned} x(k) &= Ax(k-1) + Bw(k), \quad x(0) = 0, \\ y(k) &= Cx(k-1) + Dw(k), \end{aligned} \tag{38}$$

with state x, exogenous input w and output y. We assume that A is asymptotically stable and that $w(k)$ has zero mean, unit variance, and is independent and identically distributed for all $k = 1, 2, \ldots$. Thus, the power semi-norm of the output, $||y||_\mathcal{P}$, is well defined and equivalent to the $\mathcal{H}_2$ gain of the transfer function G from w to y (see [13, p.107]).

The following well-known theorem will be used:

Theorem A.1: (Lyapunov, see e.g. [13, p.527]) Given any $Q > 0$, there exists a unique $X > 0$ satisfying $A^\mathrm{T} X A - X + Q = 0$ if and only if A is asymptotically stable. The solution X is given by

$$X = \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^\mathrm{T})^k Q A^k.$$

First, we will reformulate the output power using the observability Gramian. The observability Gramian is defined as

$$X_o := \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^\mathrm{T})^k C^\mathrm{T} C A^k, \tag{39}$$

and is therefore, according to Theorem A.1, the solution to

$$A^\mathrm{T} X_o A - X_o + C^\mathrm{T} C = 0. \tag{40}$$

The output power is well-defined, since A is asymptotically stable, and given by

$$\begin{aligned}||y||_{\mathcal{P}}^2 &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y(k)^{\mathrm{T}} y(k) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{tr}(y(k)y(k)^{\mathrm{T}}) \\ &= \left[\lim_{N\to\infty} \sum_{k=0}^{N-1} \mathrm{tr}(CA^k BB^{\mathrm{T}}(A^{\mathrm{T}})^k C^{\mathrm{T}})\right] + \mathrm{tr}(DD^{\mathrm{T}}),\end{aligned}$$

where the statistical properties of $w(k)$ have been exploited in the last step. Thus, it follows that

$$\begin{aligned}||y||_{\mathcal{P}}^2 &= \mathrm{tr}(B^{\mathrm{T}} \left\{\lim_{N\to\infty} \sum_{k=0}^{N-1} (A^{\mathrm{T}})^k C^{\mathrm{T}} C A^k \right\} B) + \mathrm{tr}(DD^{\mathrm{T}}) \\ &= \mathrm{tr}(B^{\mathrm{T}} X_o B) + \mathrm{tr}(D^{\mathrm{T}} D) \\ &= \mathrm{tr}(B^{\mathrm{T}} X_o B + D^{\mathrm{T}} D).\end{aligned}$$

Theorem A.2: It holds that

$$||y||_{\mathcal{P}}^2 = \mathrm{tr}(B^{\mathrm{T}} X_o B + D^{\mathrm{T}} D) < \gamma,$$

where X_o is the observability Gramian, if and only if there exists a symmetric, positive definite matrix X such that

$$\mathrm{tr}(B^{\mathrm{T}} X B + D^{\mathrm{T}} D) < \gamma \quad \text{and} \quad A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C < 0.$$

Proof: (Sketch) ($\Rightarrow$)

$$||y||_{\mathcal{P}}^2 = \mathrm{tr}(B^{\mathrm{T}} X_o B + D^{\mathrm{T}} D) < \gamma$$

implies that there exists an ε small enough such that

$$\mathrm{tr}(B^{\mathrm{T}} X B + D^{\mathrm{T}} D) < \gamma$$

with

$$\begin{aligned}X :&= X_o + \epsilon^2 \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^{\mathrm{T}})^k A^k \\ &= \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^{\mathrm{T}})^k (C^{\mathrm{T}} C + \epsilon^2 I) A^k.\end{aligned}$$

Theorem A.1 implies that $A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C + \epsilon^2 I = 0$ and therefore $A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C < 0$.

($\Leftarrow$) From $A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C < 0$ it follows that there exists a $Q = Q^{\mathrm{T}} > 0$ such that $A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C + Q = 0$. By definition of the observability Gramian, $C^{\mathrm{T}} C = -A^{\mathrm{T}} X_o A + X_o$ and we have that $A^{\mathrm{T}}(X - X_o)A - (X - X_o) + Q = 0$. According to Theorem A.1 the solution of this Lyapunov equation is given by $X - X_o = \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^{\mathrm{T}})^k Q A^k > 0$. Therefore $X = X_o + \lim_{N\to\infty} \sum_{k=0}^{N-1} (A^{\mathrm{T}})^k Q A^k$, which implies that

$$||y||_{\mathcal{P}}^2 = \mathrm{tr}(B^{\mathrm{T}} X_o B + D^{\mathrm{T}} D) \leq \mathrm{tr}(B^{\mathrm{T}} X B + D^{\mathrm{T}} D) < \gamma.$$

■

The previous theorem can be used to express the $\mathcal{H}_2$ norm of the system as a set of LMI conditions, that is

$$\begin{aligned}||y||_{\mathcal{P}}^2 < \gamma \quad \Leftrightarrow \quad & \mathrm{tr}(H) < \gamma, \quad H = H^{\mathrm{T}} > 0, \\ & B^{\mathrm{T}} X B + D^{\mathrm{T}} D - H < 0, \\ & A^{\mathrm{T}} X A - X + C^{\mathrm{T}} C < 0.\end{aligned}$$

Applying the Schur complement (twice) to $B^{\mathrm{T}} X B + D^{\mathrm{T}} D - H < 0$ results in

$$\begin{pmatrix} X & XB \\ B^{\mathrm{T}} X & H - D^{\mathrm{T}} D \end{pmatrix} > 0 \Leftrightarrow \begin{pmatrix} I & 0 & D \\ 0 & X & XB \\ D^{\mathrm{T}} & B^{\mathrm{T}} X & H \end{pmatrix} > 0.$$

Similarly, the matrix inequality $A^{\mathrm{T}} X A - X + CC^{\mathrm{T}} < 0$ can be reformulated as

$$\begin{pmatrix} X & XA \\ A^{\mathrm{T}} X & X - C^{\mathrm{T}} C \end{pmatrix} > 0 \Leftrightarrow \begin{pmatrix} I & 0 & C \\ 0 & X & XA \\ C^{\mathrm{T}} & A^{\mathrm{T}} X & X \end{pmatrix} > 0,$$

which establishes the conditions (32).

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